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# One-loop anisotropy for improved actions

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Abstract: We determine the one-loop correction to the anisotropy factor for the square Symanzik improved lattice action, extracted from the finite volume effective action for SU($\mathcal{N}$) gauge theories in the background of a zero-momentum gauge field. The result is smaller by approximately a factor 3 than the one-loop correction for the anisotropic Wilson action. We also comment on the Hamiltonian limit.

## 1 Introduction

Improved actions [1, 2] have become a frequently used tool for doing Monte Carlo simulations. Recently the square Symanzik improved action was introduced [3], motivated by the desire to simplify perturbative calculations. From the numerical point of view this new improved action is not expected to be more optimal in removing lattice spacing errors, although simulations [4, 5] suggest it is not performing much worse than the Lüscher-Weisz choice [2] of the improved action.

The square Symanzik action was introduced to allow for a simple background covariant gauge condition. The background field calculation is particularly suited for computing the renormalized coupling constant, not only in the continuum [6], but also on the lattice with, or without anisotropy [7, 8]. Improved anisotropic lattices are used both for thermodynamics [5] and for extracting glueball masses on very coarse lattices [9]. In both cases the aim is to enhance the resolution in the time direction. Also for the square Symanzik action anisotropy was introduced and used in Monte Carlo simulations [5]. This has motivated us to compute the one-loop correction to the anisotropy factor for this improved action, as it requires only a minor modification in the calculation already performed to compute the Lambda parameter for its isotropic version.

We will perform the background field calculation for a finite volume at arbitrary anisotropy $\xi$, using the methods followed for the isotropic Wilson action [10]. Our results will include the Wilson action and the square Symanzik improved action. The one-loop correction to the anisotropy factor for the Wilson action was computed before by Karsch [8] in an infinite volume. Our finite volume calculation nicely confirms the universality of these results.

## 2 The anisotropic square Symanzik action

For the Wilson action the anisotropy was introduced as follows ( $\underset{x\ \ \mu}{\bullet\!\!\longrightarrow} \equiv U_\mu(x) \in \mathrm{SU}(\mathcal{N})$):

$$S_W(\xi) = \frac{1}{g_0^2}\sum_x \left[\eta\xi^{-1}\sum_{i>j>0} P_{ij} + \eta^{-1}\xi\sum_{i\neq 0} P_{0i}\right],\ P_{\mu\nu} = 2\Re e\,\mathrm{Tr}\,(1 - \underset{x\ \mu}{{}_{\nu}\square}), \tag{1}$$

where ($\beta \equiv 2\mathcal{N}/g_0^2$ for $\mathrm{SU}(\mathcal{N})$)

$$\eta(\xi, g_0) = 1 + \eta_1(\xi)/\beta + \cdots \tag{2}$$

is required to guarantee that the symmetry of interchanging space and time is restored (in the infinite volume and continuum limit hence restoring Lorentz, or rather O(4), invariance). As we have changed the discretization of the theory, also the Lambda parameter belonging to the running coupling will depend on the anisotropy parameter. Both $\eta_1(\xi)$ and $\Lambda(\xi)$ can be determined from a one-loop calculation. In this paper we will use the notation $\xi(g_0) \equiv \eta^{-1}(\xi, g_0)\xi$, sometimes in the literature also denoted by $\gamma$. In the following $\xi(g_0)$ will be denoted by $\xi$ for short; from the context it should be clear when $\xi$ indicates the tree-level value.

As was formulated in ref. [5], one can similarly introduce anisotropy for a tree-level improved action,

$$S(\{c_i\}) \equiv \sum_x \Re e\,\mathrm{Tr}\sum_{\mu\neq\nu}\frac{\xi_{\mu\nu}}{g_0^2}\left\{c_0\left(1 - \underset{x\ \mu}{{}_{\nu}\square}\right) + 2c_1\left(1 - \underset{x\ \mu}{{}_{\nu}\square\!\square}\right) + c_4\left(1 - \underset{x\ \mu}{{}_{\nu}\boxed{\phantom{xx}}}\right)\right\}, \tag{3}$$

with

$$\xi_{\mu\nu} = \xi_\mu\xi_\nu, \quad \xi_i = \xi^{-\frac{1}{2}}, \quad \xi_0 = \xi^{\frac{3}{2}}. \tag{4}$$

We wish to emphasize that the issue here is not to improve this action beyond tree-level. It would involve the extra non-planar Wilson loops that also appear in the isotropic case [2]. Its coefficients, as well as the one-loop corrections to $c_0$, $c_1$ and $c_4$ will be doubled in number due to the anisotropy. After eliminating redundancies extra parameters will have to be determined, one of which can be related to $\eta_1$. However, additional physical quantities are required to fix all one-loop coefficients, making this a less than straightforward generalization from the isotropic case [2, 11, 12]. The renormalization of the anisotropy parameter is, however, determined by requiring the restoration of the space-time symmetries in the continuum limit, and can therefore be addressed without computing the one-loop corrections to the improvement coefficients.

We impose the renormalization condition not directly by the requirement to restore the space-time symmetries, but rather by comparing the finite volume effective action in a zero-momentum background field derived from the anisotropic lattice action in eq. (3) with the result for the isotropic lattice action. We may also compare with the result obtained from dimensional regularization in the continuum. We will study the one-parameter family of actions defined by

$$c_0 = 1/(1+4z)^2,\ c_1 = zc_0,\ c_4 = c_1^2/c_0, \tag{5}$$

where $z = 0$ corresponds to the Wilson action and $z = -1/16$ corresponds to the square Symanzik action, which is improved at tree-level to second order in the lattice spacing.

The relation $c_4 = c_1^2/c_0$ allows for a simple background field covariant gauge condition that is easily generalized to the anisotropic case.

$$\hat{\mathcal{F}}_{gf} \equiv \sqrt{c_0} \sum_\mu \xi_\mu \hat{D}_\mu^\dagger \left(1 + z(2 + \hat{D}_\mu^\dagger)(2 + \hat{D}_\mu)\right) \hat{q}_\mu(x) = 0. \tag{6}$$

The background covariant derivative is given by $\hat{D}_\mu \Phi(x) \equiv \hat{U}_\mu(x)\Phi(x+\hat{\mu})\hat{U}_\mu^\dagger(x) - \Phi(x)$ and the quantum fluctuations are parametrized as $U_\mu(x) = e^{\hat{q}_\mu(x)}\hat{U}_\mu(x)$ for a lattice background field $\hat{U}_\mu(x)$. The free propagators in this gauge (at $\hat{U}_\mu(x) \equiv 1$) are given by

$$\begin{aligned} \text{Ghost :} \qquad & P(k) = \frac{1}{\sqrt{c_0}\sum_\lambda \xi_\lambda \left(4\sin^2(k_\lambda/2) + 4z\sin^2 k_\lambda\right)} \ , \\ \text{Vector :} \qquad & P_{\mu\nu}(k) = \frac{P(k)\delta_{\mu\nu}}{\sqrt{c_0}\,\xi_\mu(1 + 4z\cos^2(k_\mu/2))} \ . \end{aligned} \tag{7}$$

# 3 Background field calculation

We compute on a lattice of size $N^3 \times \infty$ the effective action for a dynamical (i.e. time dependent) zero-momentum non-Abelian background field, $\hat{U}_j(x) \equiv \exp(\hat{c}_j(t)) \equiv \exp(c_j^a(t)T_a/N)$ and $\hat{U}_0(x) \equiv 1$ (the anti-hermitian generators $T_a$ are normalized as $\mathrm{Tr}\,(T_a T_b) = -\frac{1}{2}\delta_{ab}$). It is obtained by intergrating out all non-zero momentum modes. No integration over the zero-momentum quantum modes is included, which for a dynamical background field would lead to breakdown of the adiabatic approximation near $c = 0$, where the classical potential is quartic [13]. We will follow closely the methods developed for the isotropic Wilson action, described at great length before [10]. The effective action is given by

$$\sum_t a_t \left\{ \left(\frac{1}{g^2} + \alpha_1 - \frac{\eta_1}{2\mathcal{N}}\right) \frac{\left(c_i^a(t+1) - c_i^a(t)\right)^2}{2a_t^2 L^{-1}} + \frac{1}{4L}\left(\frac{1}{g^2} + \alpha_2 + \frac{\eta_1}{2\mathcal{N}}\right)\left(F_{ij}^a(t)\right)^2 + V_1(c(t)) \right\}, \tag{8}$$

where $a_t = L/\xi N$ is the lattice spacing in the time direction, $L$ the physical size of the volume, $F_{ij}^a = \varepsilon_{abe} c_i^b c_j^e$ the field strength and $V_1(c)$ is by definition the rest of the effective potential. All that is relevant to know is that at $\mathcal{O}(c^4)$ its coefficients are fixed uniquely by an abelian background field, unambiguously separating $(F_{ij}^a)^2$ from $V_1(c)$. We have ignored terms that vanish in the continuum limit. Furthermore, the renormalization group to one-loop order implies $g_0^{-2} = -11\mathcal{N}\log(a_s\Lambda)/24\pi^2$, where $a_s = L/N = a_t\xi$ is the lattice spacing in the space directions. We have therefore introduced the renormalized coupling $g^{-2} \equiv g_0^{-2} - 11\mathcal{N}\log(N)/24\pi^2 = -11\mathcal{N}\log(L\Lambda)/24\pi^2$.

We note that $\Lambda$, $\eta_1$, $\alpha_1$ and $\alpha_2$ depend on $\xi$ and $z$. Universality requires that physical quantities, as well as the background field effective action, are independent of these parameters in the continuum limit. This implies that both $(\alpha_1 - 11\mathcal{N}\log(L\Lambda)/24\pi^2 - \eta_1/2\mathcal{N})$ and $(\alpha_2 - 11\mathcal{N}\log(L\Lambda)/24\pi^2 + \eta_1/2\mathcal{N})$ are independent of $\xi$ and $z$. For isotropic lattices ($\xi = 1$), for which $\eta_1 = 0$, this implies that

$$\alpha_1 - \alpha_1^c = \alpha_2 - \alpha_2^c = 11\mathcal{N}\log(\Lambda/\Lambda^c)/24\pi^2 \quad (\xi \equiv 1), \tag{9}$$

where $\alpha_i^c$ are the values for a fixed isotropic regularization, like dimensional regularization in the continuum, or the isotropic Wilson action. For anisotropic lattices we follow Karsch [8]

by defining

$$c_\tau(\xi) \equiv \alpha_1(1) - \alpha_1(\xi), \quad c_\sigma(\xi) \equiv \alpha_2(1) - \alpha_2(\xi) \tag{10}$$

and one easily derives

$$\eta_1(\xi) = \mathcal{N}(c_\sigma(\xi) - c_\tau(\xi)), \quad \Lambda(\xi)/\Lambda(1) = \exp\left(-12\pi^2[c_\sigma(\xi) + c_\tau(\xi)]/11\mathcal{N}\right). \tag{11}$$

We can summarize these various relations between the Lambda parameters also as

$$\Lambda(\xi, z)/\Lambda(\xi', z') = \exp\left(12\pi^2[\alpha_1(\xi, z) + \alpha_2(\xi, z) - \alpha_1(\xi', z') - \alpha_2(\xi', z')]/11\mathcal{N}\right). \tag{12}$$

# 4 Analytic results

The coefficients $\alpha_1$ and $\alpha_2$ are determined by working out the determinants of the quadratic fluctuation operator. This section can be skipped when one is interested in the numerical results only. To all orders in the background field and to quadratic order in the quantum field, using $c_0^a = 0$ and $c_\mu^a(x+\hat{\mu}) = c_\mu^a(x) = c_\mu^a(t)$, we find ($k, \ell \in \{1, 2\}$)

$$\begin{aligned} S_2 = \frac{c_0}{4g_0^2}\sum_x \mathrm{Tr}\Bigg(&\sum_{\mu,\nu,k,\ell} \xi_{\mu\nu} z^{k+\ell-2}\Big\{(\hat{S}^+_{k\hat{\mu},\ell\hat{\nu}}(t) - 2)\left(\hat{D}_{k\hat{\mu}}\hat{q}_{\ell\hat{\nu}}(x) - \hat{D}_{\ell\hat{\nu}}\hat{q}_{k\hat{\mu}}(x)\right)^2 - \\ &\hat{S}^-_{k\hat{\mu},\ell\hat{\nu}}(t)\{[q_{\ell\hat{\nu}}(x) + \hat{D}_{\ell\hat{\nu}}\hat{q}_{k\hat{\mu}}(x), \hat{q}_{k\hat{\mu}}(x) + \hat{D}_{k\hat{\mu}}\hat{q}_{\ell\hat{\nu}}(x)] + [\hat{q}_{\ell\hat{\nu}}(x), \hat{q}_{k\hat{\mu}}(x)] + \\ &(\ell + k - 2)\hat{D}_{\ell\hat{\nu}}[\hat{q}_\mu(x), \hat{D}_\mu \hat{q}_\mu(x)]\}\Big\} - 4\Big\{\sum_{\mu,k}\xi_\mu z^{k-1}\hat{D}^\dagger_{k\hat{\mu}}\hat{q}_{k\hat{\mu}}(x)\Big\}^2\Bigg). \end{aligned} \tag{13}$$

We have introduced the following convenient shorthand notations

$$\begin{aligned} &\hat{S}^+_{k\hat{\mu},\ell\hat{\nu}}(t) = 2 - \hat{S}_{k\hat{\mu},\ell\hat{\nu}}(t) - \hat{S}^\dagger_{k\hat{\mu},\ell\hat{\nu}}(t), \quad \hat{S}^-_{k\hat{\mu},\ell\hat{\nu}}(t) = \hat{S}^\dagger_{k\hat{\mu},\ell\hat{\nu}}(t) - \hat{S}_{k\hat{\mu},\ell\hat{\nu}}(t), \\ &\hat{S}_{k\hat{\mu},\ell\hat{\nu}}(t) = e^{k\hat{c}_\mu(t)} e^{\ell\hat{c}_\nu(t+k\hat{\mu})} e^{-k\hat{c}_\mu(t+\ell\hat{\nu})} e^{-\ell\hat{c}_\nu(t)}, \end{aligned} \tag{14}$$

as well as "doubled" covariant derivatives and quantum fields ($q_{1\hat{\mu}}(x) \equiv q_\mu(x)$ and $D_{1\hat{\mu}} \equiv D_\mu$)

$$\hat{D}_{2\hat{\mu}}\Phi(x) = e^{2\hat{c}_\mu(t)}\Phi(x+2\hat{\mu})e^{-2\hat{c}_\mu(t)} - \Phi(x), \quad \hat{q}_{2\hat{\mu}}(x) = 2\hat{q}_\mu(x) + \hat{D}_\mu\hat{q}_\mu(x). \tag{15}$$

Therefore $\hat{D}_{2\hat{\mu}} = 2\hat{D}_\mu + \hat{D}_\mu^2$ and the gauge fixing functional can be written as

$$\hat{\mathcal{F}}_{\rm gf}(x) = \sqrt{c_0}\sum_{\mu,k}\xi_\mu z^{k-1}\hat{D}^\dagger_{k\hat{\mu}}\hat{q}_{k\hat{\mu}}(x). \tag{16}$$

$S_2$ was obtained by adding $-\sum_x \mathrm{Tr}\,\hat{\mathcal{F}}^2_{\rm gf}(x)/g_0^2$ to the action $S$. Under an infinitesimal gauge transformation, $\hat{q}_{k\hat{\mu}}(x)$ transforms as $D_{k\hat{\mu}}\Phi(x)$ for $k = 1, 2$, giving for the ghost operator

$$\mathcal{M}_{\rm gh} = \sqrt{c_0}\sum_{\mu,k}\xi_\mu z^{k-1}\hat{D}^\dagger_{k\hat{\mu}}\hat{D}_{k\hat{\mu}}. \tag{17}$$

It is now straightforward to compute the functional determinants. Simplifications can be made due to the fact that we can split the one-loop correction in a purely kinetic part for which we can drop all terms of higher than second order in $c$ and a potential term for

which the time dependence of the background field can be ignored. We find the following results

$$\alpha_{s+1}(\mathcal{X}) \equiv \alpha_d^{(s,2)}(\mathcal{X}) + \xi^2\alpha_o^{(s,2)}(\mathcal{X}) + \xi^{2s}\alpha_o^{(2,s)}(\mathcal{X}) + \alpha_b^{(s,2)}(\mathcal{X}), \tag{18}$$

with $\mathcal{X} = \{\xi, z, \mathcal{N}, N\}$, $s = 0, 1$ and

$$\begin{aligned}
\alpha_d^{(\mu,\nu)}(\mathcal{X}) &= \frac{11\mathcal{N}}{24\pi^2}\log(N) + \frac{\mathcal{N}}{6\pi\xi^3N^3}\sum_{\vec{k}\neq\vec{0}}\int_{-\pi}^{\pi} dk_0\Big(d_\mu d_\nu - 3q_\mu q_\nu\Big)P^2, \qquad (19)\\
\alpha_o^{(\mu,\nu)}(\mathcal{X}) &= \frac{\mathcal{N}}{8\pi\xi^3N^3}\sum_{\vec{k}\neq\vec{0}}\int_{-\pi}^{\pi} dk_0\Big\{2(4-3p_\nu)(\frac{\zeta-d_\mu}{\mathcal{N}^2}-\zeta) + d_\mu(\zeta p_\nu^2 + 4(1-p_\nu))\Big\}P,\\
\alpha_b^{(s,\nu)}(\mathcal{X}) &= \frac{(s+1)\mathcal{N}}{12\pi\xi N^3}\sum_{\vec{k}\neq\vec{0}}\int_{-\pi}^{\pi} dk_0\ \frac{\partial^2}{\partial k_\nu^2}\Big\{\frac{3\xi^{2s}}{4}\Big[\zeta p_s^2 + 4(1-p_s)\Big]\log P + \xi^2\frac{\partial^2\log P}{\partial k_s^2} + d_sP\Big\}\\
&\qquad + \frac{\partial}{\partial k_\nu}\left(6\zeta P\frac{\partial \log p_\nu}{\partial k_\nu}\right),
\end{aligned}$$

where $\zeta \equiv 1+4z$, $P \equiv P(k,\xi,z)$ is the (rescaled) propagator, and $d_\mu$, $p_\mu$ and $q_\mu$ are simple trigonometric functions (momenta are given as $k \equiv (k_0, \vec{k}) = (k_0, 2\pi\vec{n}/N)$, $n_i \in Z_N$)

$$\begin{aligned}
&P = \Big[4\sin^2(\tfrac{1}{2}k_0)\Big(1+4z\cos^2(\tfrac{1}{2}k_0)\Big)+\hat{\omega}^2\Big]^{-1},\ \hat{\omega}^2 = \xi^{-2}\sum_i 4\sin^2(\tfrac{1}{2}k_i)\Big(1+4z\cos^2(\tfrac{1}{2}k_i)\Big),\\
&d_\mu = \cos(k_\mu)+4z\cos(2k_\mu),\ p_\mu = [1+2z(1+\cos(k_\mu))]^{-1},\ q_\mu = p_\mu^{-1}(1+\cos(k_\mu))(2-\zeta p_\mu)^2. \qquad (20)
\end{aligned}$$

We note that in the continuum limit, $N \to \infty$, the total derivative terms $\alpha_b$ will only get contributions from near $\vec{k} = \vec{0}$. It can be shown that

$$\lim_{N\to\infty}\alpha_b^{(s,2)}(\mathcal{X}) = \mathcal{N}\left(\frac{s+1}{72\pi^2} + s\left(\frac{8a_4}{5} - \frac{1}{45\pi^2}\right)\right), \quad a_4 = -(4\pi)^{-2}\cdot 0.619331710\cdots \tag{21}$$

where $a_4$ is a constant introduced in ref. [13]. In particular these boundary contributions are independent of $\xi$ and $z$ and drop out in the computation of the quantities in eqs. (10-12). The remaining terms can be converted to *finite* integrals in the continuum limit, replacing $N^{-3}\sum_{\vec{k}}$ by $(2\pi)^{-3}\int_{-\pi}^{\pi} d^3\vec{k}$.

For isotropic ($\xi = 1$) actions, $\alpha_o^{(2,s)} = \alpha_o^{(s,2)}$ and $\alpha_d^{(s,2)}$ are independent of $s$. Consequently $\alpha_1 - \alpha_2 = \mathcal{N}(1/24\pi^2 - 8a_4)/5$ is independent of the regularization employed, cmp. eq. (9). The non-vanishing value of this difference is a manifestation of the breakdown of Lorentz invariance in a finite physical volume. At finite $N$ our results in eqs. (18-19) are exact. All integrals over $k_0$ can be performed analytically, yielding sums over the $N^3 - 1$ non-zero spatial momenta of *explicit* analytic expressions in $z$, $\xi$, $\mathcal{N}$ and $\vec{k}$, which are readily evaluated numerically. It can be shown that as an expansion in $1/N$, terms linear *and for* $z = -1/16$ *(i.e. with improvement) quadratic* in the lattice spacing are absent. For $z = 0$, where $p_\mu = \zeta = 1$, $q_\mu = (1 + d_\mu)$ and $d_\mu = \cos(k_\mu)$, dramatic simplifications occur. There are in particular for $z = 0$ more efficient ways to compute the coefficients, but we will not dwell on this any further.

We list the following $k_0$ integrals required to evaluate the expressions in eq. (19) ($y(\hat{\omega})$ is defined by $\hat{\omega} \equiv 2\sinh(\frac{1}{2}y)$ and $m > 0$)

$$\delta_m^0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat{\omega}^2}\right)^{m-1}\left\{\frac{1}{2\sinh(y)\sqrt{1+4ze^{-y}}}\right\},$$

$$
\begin{aligned}
\delta^1_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ \sin^2(k_0)P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat\omega^2}\right)^{m-1}\left\{\frac{e^{-y}}{1+4ze^{-y}+\sqrt{1+4ze^{-y}}}\right\},\\
\delta^{-1}_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ z^{-1}(1-p_0)P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat\omega^2}\right)^{m-1}\left\{4\hat\omega^{-2}\left(\frac{1}{\zeta+\sqrt{\zeta}}-\delta^1_1\right)\right\},\\
\delta^{-2}_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ p_0^2P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat\omega^2}\right)^{m-1}\left\{\frac{1+\zeta}{2\hat\omega^2\zeta^{\frac{3}{2}}}+\frac{\zeta\delta^{-1}_1-4\delta^0_1}{\hat\omega^2}\right\},\\
\mathcal{K}^1_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ d_0P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat\omega^2}\right)^{m-1}\left\{(\tfrac12\hat\omega^2+\zeta)\delta^0_1-6z\delta^1_1-\tfrac12\right\},\\
\mathcal{K}^2_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_0\ q_0P^m = \frac{1}{(m-1)!}\left(-\frac{\partial}{\partial\hat\omega^2}\right)^{m-1}\left\{\tfrac12\zeta^2\delta^{-1}_1-8z\delta^1_1\right\}.
\end{aligned}
\tag{22}
$$

# 5 Numerical results

By extrapolating in $1/N$ the explicit expressions for $\alpha_1$ and $\alpha_2$ on a finite lattice, we can extract their continuum limit to at least nine digit accuracy (we evaluate the momentum sums on lattices with $N = 3$ to $N = 99$). Our results are valid for arbitrary SU($\mathcal{N}$), where $\alpha_{1,2}$ can be written as $a\mathcal{N} + b/\mathcal{N}$. Using eqs. (10-11) we reproduce results obtained by Karsch [8] for the Wilson action. The ratio of the square Symanzik action Lambda parameter to the Wilson action Lambda parameter is obtained using either eq. (9) or eq. (12) with $(z,\xi) = (-1/16, 1)$ and $(z',\xi') = (0, 1)$. The result was already reported in ref. [3]. One finds

$$
\Lambda_{\rm sq}/\Lambda_{\rm W} = \begin{cases} 4.0919901(1) & \text{for } \mathcal{N}=2,\\ 5.2089503(1) & \text{for } \mathcal{N}=3,\end{cases}
\tag{23}
$$

agreeing with two alternative recent determinations based on the heavy-quark potential and twisted finite volume spectroscopy [12].

In table 1 we give results for the square Symanzik action at some selected values of $\xi$ between 1 and 20, likely to be of use in simulations, as well as for the Hamiltonian limit, $\xi=\infty$. The $\xi$ dependence is illustrated in figure 1. Results for the Wilson action are given for comparison. We only present $\eta_1$ and $\Lambda(\xi)/\Lambda(1)$ for SU(2) and for SU(3), since one can use eq. (11) to extract the values of $c_\tau$ and $c_\sigma$. These can furthermore be used to extract the results for any other number of colors, since

$$
c_{\tau,\sigma}(\mathcal{N}) = \frac{2(9-\mathcal{N}^2)}{5\mathcal{N}}c_{\tau,\sigma}(2)+\frac{3(\mathcal{N}^2-4)}{5\mathcal{N}}c_{\tau,\sigma}(3).
\tag{24}
$$

We note that in all cases the value of $\eta_1$ is reduced by a factor of approximately 3 for the improved square Symanzik action as compared to the result for the Wilson action. Indeed for the simulations performed in ref. [5] a reduction with approximately a factor 2.5 for the measured value of $\eta_1$ can be deduced (whereas the results obtained from the Lüscher-Weisz and square Symanzik action agree within errors). We extracted $\eta_1$ using the one-loop truncation of eq. (2). At the rather strong coupling employed in these simulations the measured values of $\eta_1$ themselves should of course not be expected to agree with the perturbative results [14].

# 6 Hamiltonian limit

Here we briefly discuss an interesting feature of the Hamiltonian limit, i.e. $\xi \to \infty$. It turns out that the potential $V_1(c)$ has field dependent contributions that diverge in this limit. This divergence, however, only occurs for $z \neq 0$, in particular for the improved square Symanzik action, i.e. at $z = -1/16$. At first sight this may seem puzzling. However one should notice that when modifying the action also the measure of integration has to be corrected. Such a modification of the measure can of course be absorbed in the action, as is usually done and is of the same order as the one-loop corrections, giving rise to a term $\sum_{t,i} \delta V(c_i(t))$. Since the effective potential appears in the action as $\sum_t a_t V_1(c(t))$, with $a_t = L/N\xi$, we conclude that the total contribution to the effective potential, due to correcting for the measure, is linear in $\xi$ (vanishing for $N \to \infty$) and given by $N\xi \sum_i \delta V(c_i)/L$. It is hence much more natural to redirect any terms linear in $\xi$ to the measure.

To determine $\delta V$ we compute $V_1(c)$ by taking an abelian background field, suitably extended to the non-Abelian sector. For SU(2) this extension is achieved by substituting $C_i = \sqrt{\sum_a c_i^a c_i^a}$ in the Abelian background link variable $\hat{U}_j = \exp(\frac{1}{2} i C_j \sigma_3/N)$. This can be generalized to arbitrary gauge groups following the methods described in ref. [15], but we will for the sake of presentation only consider the effective potential for SU(2). At $\mathcal{O}(c^6)$ there are additional terms that vanish for Abelian background fields, but they do not concern us here (and have finite limits as $\xi \to \infty$). Along the lines described in ref. [3] anisotropy is easily incorporated and one finds $V_1(c) = V_1^{\rm ab}(\vec{C}) - V_1^{\rm ab}(\vec{0})$, where

$$V_1^{\rm ab}(\vec{C}) = \frac{N\xi}{L} \sum_{\vec{n}\neq\vec{0}\in Z_N^3} \left\{ \sum_i \log(\lambda_i) + 4\mathrm{asinh}\left( \frac{1}{\sqrt{4|z|}} \sqrt{1 + 4z + \frac{\omega^2}{2\xi^2} + \frac{\omega}{\xi}\sqrt{1 + \frac{\omega^2}{4\xi^2}}} \right) \right\}, \tag{25}$$

with $\lambda_j(n_j, C_j) = 1 + 4z\cos^2((\pi n_j + \frac{1}{2}C_j)/N)$ and $\omega^2(\vec{n}, \vec{C}) = \sum_j 4\lambda_j \sin^2((\pi n_j + \frac{1}{2}C_j)/N)$ (a more detailed derivation for the isotropic case will appear elsewhere [16]). One can verify that this gives the correct continuum limit at fixed $\xi$. For the Wilson action ($z=0$) one finds $V_1^{\rm ab}(\vec{C}) = L^{-1}N\xi\sum_{\vec{n}\neq\vec{0}} 4\mathrm{asinh}(\omega/2\xi)$ (the apparent divergence for $z = 0$ can be shown to be field independent). At finite $N$ we find $\delta V(c_i) = \sum_{\vec{n}\neq\vec{0}} \log(\lambda_i(n_i, C_i)/\lambda_i(n_i, 0))$. The remainder, denoted by $\mathcal{V}_1(c)$, can easily be shown to have a finite limit for $\xi \to \infty$. We conclude that the Haar-measure, $d\hat{U}_i(c) = N^{-1}(2\pi C_i)^{-2}\sin^2(C_i/2N)\prod_a dc_i^a$, is to be corrected with a factor $\exp[-\delta V(c_i)]$, giving the *exact* measure at finite $N$ for deriving the effective Hamiltonian from improved actions. It is not too hard to show that up to exponential corrections in $N$, we have $\exp[-\delta V(c_i)] = [1 + 4z\cos^2(C_i/2N)]/(1 + 4z)$. Indeed at $z = -1/16$ the rescaled measure is flat to $\mathcal{O}(c^3/N^3)$.

It is also interesting to point out that one finds $L\mathcal{V}_1(c) = \gamma_1(c_i^a)^2 + \mathcal{O}(c^4)$, with $\gamma_1 = \gamma_1^c - 2z/N\xi\sqrt{1+4z} + \mathcal{O}(N^{-3})$. Since at $z = -1/16$ on-shell improvement should imply that spectral quantities have no $\mathcal{O}(1/N^2)$ errors, the $1/N$ correction to $\gamma_1$ can be removed by a non-local field redefinition, as was explained to some detail in ref. [17]. The field redefinition is designed to remove the next-to-nearest couplings in the time direction (not listed in eq. (8) since they are irrelevant in the continuum limit). This non-local effect disappears in the Hamiltonian limit, as it should.

# 7 Conclusions

We have calculated the one-loop correction to the anisotropy parameter for the square Symanzik action, using a zero-momentum background field calculation in a finite periodic volume. For the Wilson case we retrieved the results in an infinite volume of Karsch [8], which is a rather non-trivial check of universality, since even in the continuum the periodic boundary conditions break the Lorentz invariance. We find that the size of the one-loop correction to the anisotropy is reduced, both for SU(2) and SU(3), by approximately a factor 3 when using the square Symanzik improved action instead of the Wilson action.

We have also discussed the "Hamiltonian limit" of the zero-momentum effective theory, where the lattice spacing in the time direction is reduced to zero. We show how the integration measure is to be improved, defining the inner product on the Hilbert space involved in extracting an effective Hamiltonian from the effective action [18].


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| | $\xi$ | Square Symanzik $\eta_1$ | Square Symanzik $\Lambda(\xi)/\Lambda(1)$ | Wilson $\eta_1$ | Wilson $\Lambda(\xi)/\Lambda(1)$ |
|---|---|---|---|---|---|
| $SU(2)$ | 1.25 | 0.0268994230 | 0.9448516628 | 0.072639575 | 0.941156329 |
| | 1.50 | 0.0438407406 | 0.9030990984 | 0.120815052 | 0.895220389 |
| | 1.75 | 0.0553875971 | 0.8715577166 | 0.154865904 | 0.862024175 |
| | 2.00 | 0.0637309147 | 0.8473816328 | 0.180064348 | 0.838519956 |
| | 2.25 | 0.0700334250 | 0.8285038242 | 0.199377634 | 0.821947171 |
| | 3.00 | 0.0821772181 | 0.7913271531 | 0.236952649 | 0.796282892 |
| | 4.00 | 0.0909349958 | 0.7647074013 | 0.263881756 | 0.786799000 |
| | 5.00 | 0.0960771084 | 0.7495355534 | 0.279376900 | 0.786585170 |
| | 6.00 | 0.0994693690 | 0.7398121769 | 0.289388945 | 0.789176074 |
| | 7.00 | 0.1018786317 | 0.7330678148 | 0.296372263 | 0.792513704 |
| | 8.00 | 0.1036794982 | 0.7281198326 | 0.301513492 | 0.795887850 |
| | 9.00 | 0.1050771985 | 0.7243357505 | 0.305453471 | 0.799053830 |
| | 10.00 | 0.1061937831 | 0.7213480410 | 0.308567683 | 0.801940305 |
| | 20.00 | 0.1112064198 | 0.7083043501 | 0.322153959 | 0.819059857 |
| | $\infty$ | 0.1162101357 | 0.6957761241 | 0.335019703 | 0.843515849 |
| $SU(3)$ | 1.25 | 0.0761124472 | 0.9441552990 | 0.202232512 | 0.940150646 |
| | 1.50 | 0.1259027090 | 0.9013716023 | 0.339196380 | 0.893219710 |
| | 1.75 | 0.1609870225 | 0.8690317960 | 0.437758448 | 0.859889187 |
| | 2.00 | 0.1870579338 | 0.8443511780 | 0.511822337 | 0.837010062 |
| | 2.25 | 0.2072120386 | 0.8252080564 | 0.569337480 | 0.821587813 |
| | 3.00 | 0.2472872614 | 0.7880636199 | 0.683440912 | 0.800832184 |
| | 4.00 | 0.2772479446 | 0.7621707435 | 0.767394275 | 0.798377544 |
| | 5.00 | 0.2952330195 | 0.7477916435 | 0.816720193 | 0.804338358 |
| | 6.00 | 0.3072395475 | 0.7387537514 | 0.849056600 | 0.812081537 |
| | 7.00 | 0.3158270143 | 0.7325745051 | 0.871852492 | 0.819692104 |
| | 8.00 | 0.3222747701 | 0.7280896992 | 0.888772783 | 0.826632621 |
| | 9.00 | 0.3272941994 | 0.7246877937 | 0.901823656 | 0.832804921 |
| | 10.00 | 0.3313126485 | 0.7220187467 | 0.912193385 | 0.838252399 |
| | 20.00 | 0.3494266569 | 0.7105270954 | 0.958042934 | 0.868813879 |
| | $\infty$ | 0.3675789970 | 0.6996590739 | 1.002502899 | 0.910408485 |

Table 1: Results for $\eta_1$, the one-loop correction to the anisotropy $\xi$, and the Lambda ratios for SU(2) and SU(3) Wilson and square Symanzik improved lattice actions.

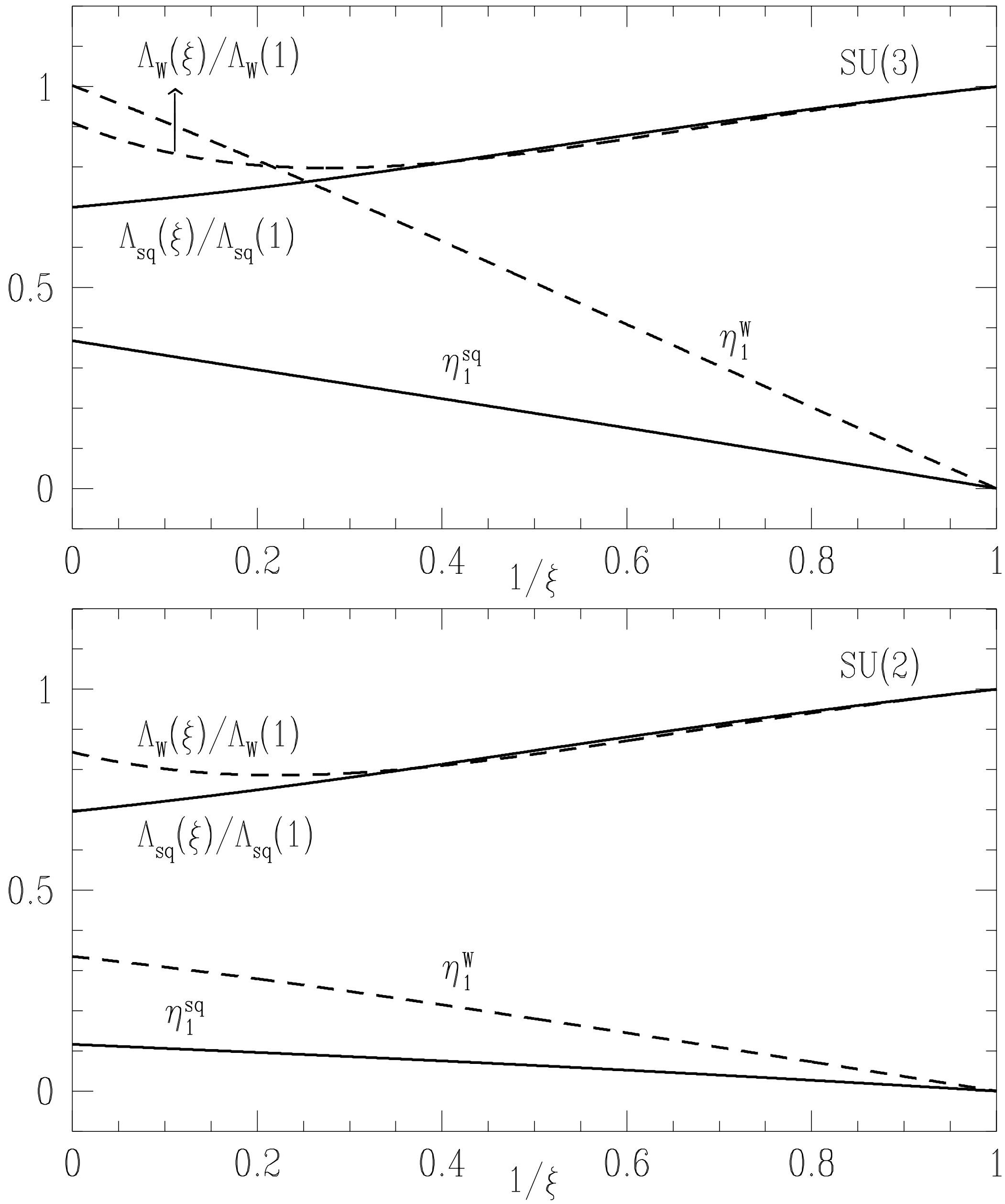


Figure 1: Comparison between the square Symanzik (sq) and Wilson (W) action results for $\eta_1$ and $\Lambda(\xi)/\Lambda(1)$. Bottom figure for SU(2) and top for SU(3).